Lecture 13:
Constrained Rigid Body Systems

Fundamentals of Computer Graphics
Animation & Simulation
Stanford CS248B, Fall 2022
3x

dt=0.01
Collision is detected! What now?

- Collision detector is responsible for returning a list of collisions at every time step.
- If the list is not empty, collision handler will take over and resolve the collisions.
- For each collision on the list, it should contain:
  - IDs of a pair of rigid bodies in collision
  - Coordinate of the contact point
  - Normal vector at the contact point
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Contact Points

Collision handler tells us that a point on A and a point on B are in collision
Contact Points

Collision handler tells us that a point on A and a point on B are in collision

Although $p_a$ and $p_b$ are coincident at time $t_c$, the velocity of the two points may be different!

$A$

$B$

Put in the world space…

$n$

$p_a(t_c) = p_b(t_c)$

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Velocity of a Contact Point

\[
\dot{p}_a(t_c) = v_a(t_c) + \omega_a(t_c) \times (p_a(t_c) - x_a(t_c))
\]

\[
\dot{p}_b(t_c) = v_b(t_c) + \omega_b(t_c) \times (p_b(t_c) - x_b(t_c))
\]

\[
v_r = \hat{n} \cdot (\dot{p}_a(t_c) - \dot{p}_b(t_c))
\]

\(v_r\) is the magnitude of the relative velocity in the normal direction
Relative Normal Velocity

\[ \mathbf{v}_r > 0 \quad \text{separation} \]

\[ \mathbf{v}_r = 0 \quad \text{resting contact} \]

\[ \mathbf{v}_r < 0 \quad \text{colliding contact} \]
Relative Normal Velocity

\( v_r > 0 \)  
**separation**

\( \hat{n} \)

\( \hat{n} \)

\( p_a(t_c) = p_b(t_c) \)

\( B \)

\( A \)

\( v_r = 0 \)  
**resting contact**

\( \hat{n} \)

\( \hat{n} \)

\( p_a(t_c) = p_b(t_c) \)

\( B \)

\( A \)

\( v_r < 0 \)  
**colliding contact**

\( \hat{n} \)

\( \hat{n} \)

\( p_a(t_c) = p_b(t_c) \)

\( B \)

\( A \)
Collision Process

\[ J \equiv \int_{0}^{\Delta t} f_t \, dt = m\Delta v \]
A Soft Collision

**force**

\[ J = \int_0^{\Delta t} f_t \, dt \]

**velocity**

\[ J = m\Delta v \]
A Hard Collision

\[ \mathbf{J} = \int_{0}^{\Delta t} \mathbf{f}_t \, dt \]

\[ \mathbf{J} = m\Delta \mathbf{v} \]
An Infinitely Hard Collision

\[ \Delta t = 0 \]

\[ J = ? \]

\[ J = m \Delta v \]
**Impulse**

- In the rigid body world, we want the velocity to change instantaneously if there is a collision contact.
- Use finite impulse to change velocity instead of infinite force: $J = \Delta P = m\Delta v$
- If the impulse acts on a point $p$, the impulse produces an impulsive torque
  \[- \tau_{imp} = (p - x(t)) \times J \]
  - Impulsive torque results in a change in angular momentum: $\tau_{imp} = \Delta L$
Colliding Contact

- For frictionless bodies, the direction of the impulse will be in the normal direction $\hat{n}(t_c)$. 

- If we solve for $j$, we then can update the linear momentum of the rigid body after the collision.

- Body A is subject to impulse $\mathbf{J}$, while B is subject to an equal but opposite impulse $-\mathbf{J}$.
Colliding Contact

- Similarly, we use impulsive torque to update the angular moment of the rigid bodies

\[ \mathbf{J} = \hat{\mathbf{n}}(t_c) \]

\[ \mathbf{j} = (\mathbf{p} - \mathbf{x}_a) \times \hat{\mathbf{n}}(t_c) \]

\[ - (\mathbf{p} - \mathbf{x}_b) \times \hat{\mathbf{n}}(t_c) \]

How to solve \( j \)?
Colliding Contact

- The change of velocity at the contact point follows the empirical law:
  \[ v_r^+ = - \epsilon v_r^- \]
- Coefficient of restitution
  - \( \epsilon = 0 \), resting contact
  - \( \epsilon = 1 \), perfect bounce

We need to solve for \( j \) such that \( v_r^+ = - \epsilon v_r^- \)
Colliding Contact

before collision

\[ \mathbf{v}_r^- = \hat{n}(t_c) \cdot (\dot{\mathbf{p}}_a^- - \dot{\mathbf{p}}_b^-) \]

after collision

\[ \mathbf{v}_r^+ = \hat{n}(t_c) \cdot (\dot{\mathbf{p}}_a^+ - \dot{\mathbf{p}}_b^+) \]

\[ \mathbf{J} = j\hat{n}(t_c) \]

\[ (\mathbf{p} - \mathbf{x}_a) \times j\hat{n}(t_c) \]

\[ -(\mathbf{p} - \mathbf{x}_b) \times j\hat{n}(t_c) \]

\[ j\hat{n}(t_c) \]
Compute the Impulse

- Define the displacement from center of mass
  - \( r_a = p_a - x_a \)
  - \( r_b = p_b - x_b \)

- Express contact point velocity in rigid body velocity
  - \( \dot{p}_a^- = v_a^- + \omega_a^- \times r_a, \) similar for \( \dot{p}_b^- \)
  - \( \dot{p}_a^+ = v_a^+ + \omega_a^+ \times r_a, \) similar for \( \dot{p}_b^+ \)

- Express post-collision velocity in unknown impulse
  - \( v_a^+ = v_a^- + \frac{j\hat{n}}{m_a}, \) similar for \( v_b^+ \)
  - \( \omega_a^+ = \omega_a^- + I_a^{-1}(r_a \times j\hat{n}), \) similar for \( \omega_b^+ \)
Compute the Impulse

- Define the displacement from center of mass
  - \( \mathbf{r}_a = \mathbf{p}_a - \mathbf{x}_a \)
  - \( \mathbf{r}_b = \mathbf{p}_b - \mathbf{x}_b \)

- Express contact point velocity in rigid body velocity
  - \( \mathbf{\dot{p}}_a^- = \mathbf{v}_a^- + \mathbf{\omega}_a^- \times \mathbf{r}_a, \) similar for \( \mathbf{\dot{p}}_b^- \)
  - \( \mathbf{\dot{p}}_a^+ = \mathbf{v}_a^+ + \mathbf{\omega}_a^+ \times \mathbf{r}_a, \) similar for \( \mathbf{\dot{p}}_b^+ \)

- Express post-collision velocity in unknown impulse
  - \( \mathbf{v}_a^+ = \mathbf{v}_a^- + \frac{\mathbf{\hat{n}}}{m_a}, \) similar for \( \mathbf{v}_b^+ \)
  - \( \mathbf{\omega}_a^+ = \mathbf{\omega}_a^- + \mathbf{I}_a^{-1}(\mathbf{r}_a \times \mathbf{\hat{n}}), \) similar for \( \mathbf{\omega}_b^+ \)

Substitute post-collision rigid body velocity
Compute the Impulse

- Define the displacement from center of mass
  - \( r_a = p_a - x_a \)
  - \( r_b = p_b - x_b \)

- Express contact point velocity in rigid body velocity
  - \( \dot{p}_a^- = v_a^- + \omega_a^- \times r_a \), similar for \( \dot{p}_b^- \)
  - \( \dot{p}_a^+ = v_a^+ + \omega_a^+ \times r_a \), similar for \( \dot{p}_b^+ \)

- Express post-collision velocity in unknown impulse
  - \( v_a^+ = v_a^- + \frac{j\hat{n}}{m_a} \), similar for \( v_b^+ \)
  - \( \omega_a^+ = \omega_a^- + I_a^{-1}(r_a \times j\hat{n}) \), similar for \( \omega_b^+ \)

Substitute post-collision rigid body velocity
Compute the Impulse

- Define the displacement from center of mass
  
  \[ r_a = p_a - x_a \]
  
  \[ r_b = p_b - x_b \]

- Express contact point velocity in rigid body velocity
  
  \[ \dot{p}_a^- = v_a^- + \omega_a^- \times r_a, \text{ similar for } \dot{p}_b^- \]
  
  \[ \dot{p}_a^+ = v_a^+ + \omega_a^+ \times r_a, \text{ similar for } \dot{p}_b^+ \]

- Express post-collision velocity in unknown impulse
  
  \[ v_a^+ = v_a^- + \frac{j\hat{n}}{m_a}, \text{ similar for } v_b^+ \]
  
  \[ \omega_a^+ = \omega_a^- + I_a^{-1}(r_a \times j\hat{n}), \text{ similar for } \omega_b^+ \]

Substitute post-collision rigid body velocity

\[ \dot{p}_a^+ = v_a^- + \frac{j\hat{n}}{m_a} + (\omega_a^- + I_a^{-1}(r_a \times j\hat{n})) \times r_a \]
Compute the Impulse

- Define the displacement from center of mass
  - \( \mathbf{r}_a = \mathbf{p}_a - \mathbf{x}_a \)
  - \( \mathbf{r}_b = \mathbf{p}_b - \mathbf{x}_b \)

- Express contact point velocity in rigid body velocity
  - \( \dot{\mathbf{p}}_a^- = \dot{\mathbf{v}}_a^- + \mathbf{\omega}_a^- \times \mathbf{r}_a , \text{similar for } \dot{\mathbf{p}}_b^- \)
  - \( \dot{\mathbf{p}}_a^+ = \dot{\mathbf{v}}_a^+ + \mathbf{\omega}_a^+ \times \mathbf{r}_a , \text{similar for } \dot{\mathbf{p}}_b^+ \)

- Express post-collision velocity in unknown impulse
  - \( \dot{\mathbf{v}}_a^+ = \dot{\mathbf{v}}_a^- + \frac{j\hat{n}}{m_a} , \text{similar for } \dot{\mathbf{v}}_b^+ \)
  - \( \mathbf{\omega}_a^+ = \mathbf{\omega}_a^- + I_a^{-1}(\mathbf{r}_a \times j\hat{n}) , \text{similar for } \mathbf{\omega}_b^+ \)

- Substitute post-collision rigid body velocity
  \[
  \dot{\mathbf{p}}_a^+ = \dot{\mathbf{v}}_a^- + \frac{j\hat{n}}{m_a} + (\mathbf{\omega}_a^- + I_a^{-1}(\mathbf{r}_a \times j\hat{n})) \times \mathbf{r}_a
  \]

- Recover pre-collision contact velocity, \( \dot{\mathbf{p}}_a^- \)
  \[
  \dot{\mathbf{p}}_a^- = \dot{\mathbf{p}}_a^+ - j \left( \frac{j\hat{n}}{m_a} + (I_a^{-1}(\mathbf{r}_a \times j\hat{n})) \times \mathbf{r}_a \right)
  \]
Compute the Impulse

- Express the empirical law in contact velocity

\[ \nu_r^+ = - \epsilon \nu_r^- \]

\[ \dot{p}_a^+ = \dot{p}_a^- + j \left( \frac{j \hat{n}}{m_a} + \left( I_a^{-1} (r_a \times j \hat{n}) \right) \times r_a \right) \]
Compute the Impulse

Express the empirical law in contact velocity

\[ v_r^+ = -\epsilon v_r^- \]

\[ v_r^+ = \hat{n} \cdot (\dot{p}_a^+ - \dot{p}_b^+) \]

\[ = \hat{n} \cdot (\dot{p}_a^- - \dot{p}_b^-) + j\left( \frac{1}{m_a} + \frac{1}{m_b} + \hat{n} \cdot (I_a^{-1}(r_a \times \hat{n})) \times r_a + \hat{n} \cdot (I_b^{-1}(r_b \times \hat{n})) \times r_b \right) \]

\[ \dot{p}_a^+ = \dot{p}_a^- + j\left( \frac{j\hat{n}}{m_a} + (I_a^{-1}(r_a \times j\hat{n})) \times r_a \right) \]
Compute the Impulse

- Express the empirical law in contact velocity

\[ v_r^+ = -\epsilon v_r^- \]

\[ v_r^+ = \hat{n} \cdot (\dot{p}_a^+ - \dot{p}_b^+) \]

\[ = \hat{n} \cdot (\dot{p}_a^- - \dot{p}_b^-) + j\left( \frac{1}{m_a} + \frac{1}{m_b} + \hat{n} \cdot (I_a^{-1}(r_a \times \hat{n})) \times r_a + \hat{n} \cdot (I_b^{-1}(r_b \times \hat{n})) \times r_b \right) \]

\[ = v_r^- + j\left( \frac{1}{m_a} + \frac{1}{m_b} + \hat{n} \cdot (I_a^{-1}(r_a \times \hat{n})) \times r_a + \hat{n} \cdot (I_b^{-1}(r_b \times \hat{n})) \times r_b \right) \]

\[ -\epsilon v_r^- = v_r^- + j\left( \frac{1}{m_a} + \frac{1}{m_b} + \hat{n} \cdot (I_a^{-1}(r_a \times \hat{n})) \times r_a + \hat{n} \cdot (I_b^{-1}(r_b \times \hat{n})) \times r_b \right) \]

\[ j = \frac{-(1 + \epsilon)v_r^-}{\frac{1}{m_a} + \frac{1}{m_b} + \hat{n} \cdot (I_a^{-1}(r_a \times \hat{n})) \times r_a + \hat{n} \cdot (I_b^{-1}(r_b \times \hat{n})) \times r_b} \]
Colliding Contact

Apply change in momentum to current state:

- Body A:
  - $P(t_c + h) = P(t_c) + J$
  - $L(t_c + h) = L(t_c) + (p - x_a) \times J$

- Body B:
  - $P(t_c + h) = P(t_c) - J$
  - $L(t_c + h) = L(t_c) + (p - x_b) \times (-J)$

after collision
Relative Normal Velocity

\[ v_r > 0 \]
\[ v_r = 0 \]
\[ v_r < 0 \]

separation
resting contact
colliding contact

\[ \hat{n} \]
\[ \mathbf{p}_a(t_c) = \mathbf{p}_b(t_c) \]
Resting Contact

- In this case, all n contact points have the zero relative velocity.
- At each contact point there is some force $f_i \hat{n}_i$, where $f_i$ is an unknown scalar and $\hat{n}_i$ is a defined normal at that contact point.
- Our goal is to determine what each $f_i$ is by solving all of them simultaneously.
- What are the conditions for $f_i$?
Non-penetration

- Let’s define penetration:
  \[ d_i = \hat{n} \cdot (p_a - p_b) \]

- \( d_i(t) > 0 \)
- \( d_i(t) = 0 \)
- \( d_i(t) < 0 \)
Non-penetration

- Let’s define penetration:
  \[ d_i = \hat{n} \cdot (p_a - p_b) \]
- We want to avoid \( d_i < 0 \)

\[ d_i(t) > 0 \]
\[ d_i(t) = 0 \]
\[ d_i(t) < 0 \]
Non-penetration

- Let’s define penetration:
  \[ d_i = \hat{n} \cdot (p_a - p_b) \]
- We want to avoid \( d_i < 0 \)
- Since collision is detected, \( d_i(t) = 0 \)

\[ d_i(t) = 0 \]
Non-penetration

- Let’s define penetration:
  \[ d_i = \mathbf{n} \cdot (\mathbf{p}_a - \mathbf{p}_b) \]
- We want to avoid \( d_i < 0 \)
- Since collision is detected, \( d_i(t) = 0 \)
- What about \( \dot{d}_i(t) \)?

\[
\dot{d}_i(t) = \dot{\mathbf{n}}_i(t) \cdot (\mathbf{p}_a(t) - \mathbf{p}_b(t)) + \mathbf{n}_i(t) \cdot (\dot{\mathbf{p}}_a(t) - \dot{\mathbf{p}}_b(t))
\]

\[
\dot{d}_i(t) = v_r = 0 \text{ because it is a resting contact}
\]
Non-penetration

- Let's define penetration:
  \[ d_i = \hat{n} \cdot (p_a - p_b) \]

- We want to avoid \( d_i < 0 \)

- At rest contact, \( d_i(t) = 0 \) and \( \dot{d}_i(t) = 0 \)
Non-penetration

- Let’s define penetration:
  \[ d_i = \hat{n} \cdot (p_a - p_b) \]
- We want to avoid \( d_i < 0 \)
- At rest contact, \( d_i(t) = 0 \) and \( \dot{d}_i(t) = 0 \)
- If \( \ddot{d}(t) < 0 \), bodies have an acceleration toward each other and the penetration will occur.
Non-penetration

- Let’s define penetration:
  \[ d_i = \hat{n} \cdot (p_a - p_b) \]

- We want to avoid \( d_i < 0 \)

- At rest contact, \( d_i(t) = 0 \) and \( \dot{d}_i(t) = 0 \)

- If \( \ddot{d}(t) < 0 \), bodies have an acceleration toward each other and the penetration will occur.

- Therefore, the first condition is \( \ddot{d}(t) \geq 0 \)
Repulsive force

- The contact forces can push bodies apart, but can never act like “glue” and hold bodies together.
- Therefore, each contact force must act outward: $f_i \geq 0$
Workless force

- The contact force at a contact point becomes zero if the bodies begin to separate.
- If contact is breaking, that is, $\ddot{d}_i(t) > 0$, then $f_i$ should be zero.
- If $f_i$ is not zero, then the contact is not breaking, that is, $\ddot{d}_i(t) = 0$.
- What is the equation that satisfies these two conditions?

$$f_i \ddot{d}_i(t) = 0$$
Compute contact forces

- Non-penetration
  \[ \ddot{d}_i(t) \geq 0 \]

- Repulsive force
  \[ f_i \geq 0 \]

- Workless force
  \[ f_i \ddot{d}_i(t) = 0 \]
Compute contact forces

- **Non-penetration**
  \[ \ddot{d}_i(t) \geq 0 \]

- **Repulsive force**
  \[ f_i \geq 0 \]

- **Workless force**
  \[ f_i \dot{d}_i(t) = 0 \]

Express \( \ddot{d}'s \) in terms of \( f'\)'s:

\[
\ddot{d}_i = \hat{n} \cdot (\ddot{p}_a - \ddot{p}_b) + 2 \hat{n} \cdot (\dot{p}_a - \dot{p}_b)
= a_{i1}f_1 + a_{i2}f_2 + \cdots + a_{in}f_n + b_i
\]

Factor out the terms that depend on \( f_j \) and assign them to \( a_{ij} \)

Assign the rest of terms to \( b_i \)

Collect all the \( a_{ij} \) to form matrix \( A \) and all the \( b_i \) to form vector \( b \)

\[
\ddot{d} = Af + b, \text{ where } \ddot{d} = [\ddot{d}_1, \ldots \ddot{d}_n] \text{ and } f = [f_1, \ldots, f_n]
\]

See details in Baraff and Witkin's course notes
Linear complementarity program (LCP)

- Solve for $\mathbf{f} = [f_1, f_2, \ldots, f_n]$

- Subject to

$$\mathbf{A}\mathbf{f} + \mathbf{b} \geq 0$$

$$\mathbf{f} \geq 0$$

$$(\mathbf{A}\mathbf{f} + \mathbf{b})^T \mathbf{f} = 0$$

Can solve it as a Quadratic Program
Solve LCP iteratively

A typical LCP:

\[
\begin{align*}
Ax + b & \geq 0 \\
x & \geq 0 \\
x^T(Ax + b) & = 0
\end{align*}
\]

\[
\begin{align*}
(M + N)x + b & \geq 0 \\
x & \geq 0 \\
x^T((M + N)x + b) & = 0
\end{align*}
\]

Fixed point iteration: update \( x \) iteratively

\[
\begin{align*}
Mx_{k+1} + Nx_k + b & \geq 0 \\
x_{k+1} & \geq 0 \\
x_{k+1}^T(Mx_{k+1} + Nx_k + b) & = 0
\end{align*}
\]

Let \( c_k \equiv Nx_k + b \)

\[
\begin{align*}
Mx_{k+1} + c_k & \geq 0 \\
x_{k+1} & \geq 0 \\
x_{k+1}^T(Mx_{k+1} + c_k) & = 0
\end{align*}
\]
Solve LCP iteratively

A typical LCP:
\[
\begin{align*}
Ax + b & \geq 0 \\
(M + N)x + b & \geq 0 \\
x & \geq 0 \\
(Ax + b)^T & = 0 \\
((M + N)x + b)^T & = 0
\end{align*}
\]

Fixed point iteration: update \( x \) iteratively
\[
Mx_{k+1} + Nx_k + b \geq 0 \\
x_{k+1} \geq 0 \\
(Mx_{k+1} + Nx_k + b)^T = 0
\]

Let \( c_k \equiv Nx_k + b \)
\[
Mx_{k+1} + c_k \geq 0 \\
x_{k+1} \geq 0 \\
(Mx_{k+1} + c_k)^T = 0
\]

Projected Gauss Seidel (PGS)

\[
M = \begin{bmatrix} a_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{nn} \end{bmatrix} \\
N = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}
\]

for \( k = 0 \) to max_iter:
\[
\begin{align*}
x_{k+1}(1) & \cdot (a_{11}x_{k+1}(1) + c_k(1)) = 0 \\
x_{k+1}(1) & = \max(0, -\frac{c_k(1)}{a_{11}}) \\
x_{k+1}(2) & \cdot (a_{21}x_{k+1}(1) + a_{22}x_{k+1}(2) + c_k(2)) = 0 \\
x_{k+1}(2) & = \max(0, -\frac{a_{21}x_{k+1}(1) + c_k(2)}{a_{22}})
\end{align*}
\]
Solve LCP iteratively

Projected Jacobi:

$$M = \begin{bmatrix} a_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{bmatrix}, \quad N = \begin{bmatrix} \alpha \cdot a_{ij} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha \cdot a_{ij} \end{bmatrix}$$

Projected Successive Over Relaxation:

$$M = \begin{bmatrix} a_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{bmatrix}, \quad N = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Projected Gauss Seidel (PGS)

for $k = 0$ to max_iter:

$$x_{k+1}(1) \cdot (a_{11}x_{k+1}(1) + c_k(1)) = 0$$

$$x_{k+1}(1) = \max(0, -\frac{c_k(1)}{a_{11}})$$

$$x_{k+1}(2) \cdot (a_{21}x_{k+1}(1) + a_{22}x_{k+1}(2) + c_k(1)) = 0$$

$$x_{k+1}(2) = \max(0, -\frac{a_{21}x_{k+1}(1) + c_k(2)}{a_{22}})$$
Velocity-based LCP

- Non-penetration

\[ \dot{d}_i(t) \geq 0 \]

- Repulsive force

\[ f_i \geq 0 \]

- Workless force

\[ f_i \dot{d}_i(t) = 0 \]
Velocity-based LCP

- Non-penetration
  \[ \frac{\partial d_i}{\partial q} \dot{q} \geq 0 \]

- Repulsive force
  \[ f_i \geq 0 \]

- Workless force
  \[ f_i \frac{\partial d}{\partial q} \dot{q} = 0 \]

General representation of configurations of two rigid bodies
\[ q = [x_a, R_a, x_b, R_b] \]

Shortest distance between two rigid bodies
\[ d(q) \]

Time derivative of \( d(q(t)) \)
\[ \dot{d}_i(q(t)) = \frac{\partial d_i}{\partial q} \dot{q} \geq 0 \]
Velocity-based LCP

- Non-penetration
  \[ \frac{\partial d_i}{\partial \dot{q}} \dot{q} \geq 0 \]

- Repulsive force
  \[ f_i \geq 0 \]

- Workless force
  \[ f_i \frac{\partial d}{\partial q} \dot{q} = 0 \]

General representation of configurations of two rigid bodies
\[ q = [x_a, R_a, x_b, R_b] \]

Shortest distance between two rigid bodies
\[ d(q) \]

Time derivative of \( d(q(t)) \)
\[ \dot{d}_i(q(t)) = \frac{\partial d_i}{\partial q} \dot{q} \geq 0 \]

Compact expression of LCP:
\[ 0 \leq f \perp \frac{\partial d}{\partial q} \dot{q} \geq 0 \]
Velocity-based LCP

- Non-penetration
  \[ \frac{\partial d_i}{\partial q} \dot{q} \geq 0 \]

- Repulsive force
  \[ f_i \geq 0 \]

- Workless force
  \[ f_i \frac{\partial d}{\partial q} \dot{q} = 0 \]

Make it implicit

\[ 0 \leq f \perp \frac{\partial d}{\partial q} \dot{q}^+ \geq 0 \]

\[ \dot{q}^+ = \dot{q}^- + M^{-1} \left( \frac{\partial d}{\partial q} \right)^T f \]

Combine with colliding case

\[ 0 \leq f \perp \frac{\partial d}{\partial q} \dot{q}^+ \geq -\epsilon \frac{\partial d}{\partial q} \dot{q}^- \]

Compact expression of LCP:

\[ 0 \leq f \perp \frac{\partial d}{\partial q} \dot{q} \geq 0 \]
Friction

- **Coulomb's Law of Friction**
  - If sliding, the kinetic friction is
    \[ f_\parallel = -\mu_k |f_\perp| \frac{v_\parallel}{|v_\parallel|} \]
  - If static, stay static as long as
    \[ |f_\parallel| \leq \mu_s |f_\perp| \]
## Friction coefficient

<table>
<thead>
<tr>
<th>Materials</th>
<th>Static Friction, $\mu_s$</th>
<th>Kinetic/Sliding Friction, $\mu_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Dry and clean</td>
<td>Lubricated</td>
</tr>
<tr>
<td>Aluminium</td>
<td>Steel</td>
<td>0.61$^{[25]}$</td>
</tr>
<tr>
<td>Aluminium</td>
<td>Aluminium</td>
<td>1.05–1.35$^{[25]}$</td>
</tr>
<tr>
<td>Gold</td>
<td>Gold</td>
<td>2.5$^{[26]}$</td>
</tr>
<tr>
<td>Platinum</td>
<td>Platinum</td>
<td>1.2$^{[25]}$</td>
</tr>
<tr>
<td>Silver</td>
<td>Silver</td>
<td>1.4$^{[25]}$</td>
</tr>
<tr>
<td>Alumina ceramic</td>
<td>Silicon nitride ceramic</td>
<td>0.04–0.05$^{[28]}$</td>
</tr>
<tr>
<td>BAM (Ceramic alloy AlMgB$_{14}$)</td>
<td>Titanium boride (TiB$_2$)</td>
<td>0.02$^{[28]}$</td>
</tr>
<tr>
<td>Brass</td>
<td>Steel</td>
<td>0.35–0.51$^{[25]}$</td>
</tr>
<tr>
<td>Cast iron</td>
<td>Copper</td>
<td>1.05$^{[25]}$</td>
</tr>
<tr>
<td>Cast iron</td>
<td>Zinc</td>
<td>0.85$^{[25]}$</td>
</tr>
<tr>
<td>Concrete</td>
<td>Rubber</td>
<td>1.0</td>
</tr>
<tr>
<td>Concrete</td>
<td>Wood</td>
<td>0.62$^{[25][31]}$</td>
</tr>
<tr>
<td>Copper</td>
<td>Glass</td>
<td>0.6$^{[25]}$</td>
</tr>
<tr>
<td>Copper</td>
<td>Steel</td>
<td>0.53$^{[32]}$</td>
</tr>
<tr>
<td>Glass</td>
<td>Glass</td>
<td>0.9–1.0$^{[25][32]}$</td>
</tr>
<tr>
<td>Human synovial fluid</td>
<td>Human cartilage</td>
<td>0.01$^{[33]}$</td>
</tr>
<tr>
<td>Ice</td>
<td>Ice</td>
<td>0.02–0.08$^{[34]}$</td>
</tr>
<tr>
<td>Polythene</td>
<td>Steel</td>
<td>0.2$^{[25][34]}$</td>
</tr>
<tr>
<td>PTFE (Teflon)</td>
<td>PTFE (Teflon)</td>
<td>0.04$^{[25][34]}$</td>
</tr>
<tr>
<td>Steel</td>
<td>Ice</td>
<td>0.03$^{[34]}$</td>
</tr>
<tr>
<td>Steel</td>
<td>PTFE (Teflon)</td>
<td>0.04$^{[25][34]}$</td>
</tr>
<tr>
<td>Steel</td>
<td>Steel</td>
<td>0.70$^{[35]}$</td>
</tr>
<tr>
<td>Wood</td>
<td>Metal</td>
<td>0.2–0.6$^{[25][31]}$</td>
</tr>
<tr>
<td>Wood</td>
<td>Wood</td>
<td>0.25–0.62$^{[25][31]}$</td>
</tr>
</tbody>
</table>
A block is pushed by an increasing horizontal force. The friction force overtime looks like: